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EXPANSIONS OF KAMPÉ DE FERIET'S DOUBLE HYPERGEOMETRIC FUNCTION OF HIGHER ORDER

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ABSTRACT

Expansions of Kampé de Feriet's double hypergeometric function of higher order (i.e. with more parameters) in two variables are obtained. By specializing the number of the parameters in this function, new and known expansions of the four double hypergeometric functions of two variables are deduced.

EXPANSIONS OF KAMPÉ DE FERIET'S

DOUBLE HYPERGEOMETRIC FUNCTION OF HIGHER ORDER

F. M. Ragab

1. In a previous paper [1] in the Crelle journal I obtained expansions of Kampe de Feriet's function of higher order (i.e. with more parameters) in two variables. It is defined by the following definitions

$$F\left(\begin{array}{c|l|c} \mu & \alpha_1, \ldots, \alpha_\mu & \\ \nu & \beta_1, \beta_1'; \ldots; \beta_\nu, \beta_\nu' & \\ \rho & \gamma_1, \ldots, \gamma_\rho & x, y \\ \sigma & \delta_1, \delta_1'; \ldots; \delta_\sigma, \delta_\sigma' & \end{array}\right) =$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\mu}(\alpha_j; m+n) \prod_{j=1}^{\nu}\{(\beta_j; m)(\beta_j'; n)\}}{\prod_{j=1}^{\rho}(\gamma_j; m+n) \prod_{j=1}^{\sigma}\{(\delta_j; m)(\delta_j'; n)\}} \frac{x^m y^n}{m!n!} \quad . \tag{1}$$

where $\mu + \nu \leq \rho + \sigma + 1$ and $(a;r) = \Gamma(a+r)/\Gamma(a)$.

For the definitions and properties of this function the reader is referred to the work of Appell and Kampe de Feriet's [2], [3] and [4]. For special values of the parameters, the function (1) reduces to the four double hypergeometric functions of two variables. So we have

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$$F\left(\begin{array}{c|c|c} 1 & \alpha & \\ 1 & \beta\,,\beta' & \\ 1 & \gamma & x,y \\ 0 & \cdots & \end{array}\right) = F^{[1]}[\alpha;\beta\,,\beta';\gamma;x,y]\,; \tag{2}$$

$$F\left(\begin{array}{c|c|c} 1 & \alpha & \\ 1 & \beta\,,\beta' & \\ 0 & \cdots & x,y \\ 1 & \delta\,,\delta' & \end{array}\right) = F^{[2]}[\alpha;\beta\,,\beta';\delta\,,\delta';x,y]\,; \tag{3}$$

$$F\left(\begin{array}{c|c|c} 0 & \cdots\cdots\cdots & \\ 2 & \beta_1,\beta_1';\beta_2,\beta_2' & \\ 1 & \gamma & x,y \\ 0 & \cdots\cdots\cdots & \end{array}\right) = F^{[3]}[\beta_1,\beta_1';\beta_2,\beta_2';\gamma;x,y]\,; \tag{4}$$

$$F\left(\begin{array}{c|c|c} 2 & \alpha_1,\alpha_2 & \\ 0 & \cdots\cdots & \\ 0 & \cdots\cdots & x,y \\ 1 & \delta_1,\delta_1' & \end{array}\right) = F^{[4]}[\alpha_1,\alpha_2;\delta_1,\delta_1';x,y]\ . \tag{5}$$

where $F^{[1]}, F^{[2]}, F^{[3]}$ and $F^{[4]}$ are Appell's functions (see [4], p. 14).

Also it is easily seen that:

$$F\left(\begin{array}{c|c|c}\mu & a_1,\ldots,a_\mu & \\ 0 & \cdots\cdots & \\ \rho & \gamma_1,\ldots,\gamma_\rho & \\ \sigma & \cdots\cdots & \end{array}\; x, y\right) = {}_\mu F_\rho\left(\begin{array}{l} a_1,\ldots,a_\mu ; x+y \\ \gamma_1,\ldots,\gamma_\rho \end{array}\right); \qquad (6)$$

$$F\left(\begin{array}{c|c|c}0 & \cdots\cdots\cdots & \\ \nu & \beta_1,\beta_1';\ldots;\beta_\nu,\beta_\nu' & \\ 0 & \cdots\cdots\cdots & \\ \sigma & \delta_1,\delta_1';\ldots;\delta_\sigma,\delta_\sigma' & \end{array}\; x, y\right) = {}_\nu F_\sigma\left(\begin{array}{l} \beta_1,\ldots,\beta_\nu ; x \\ \delta_1,\ldots,\delta_\sigma \end{array}\right) {}_\nu F_\sigma\left(\begin{array}{l} \beta_1',\ldots,\beta_\nu' ; y \\ \delta_1',\ldots,\delta_\sigma' \end{array}\right); \qquad (7)$$

$$F\left(\begin{array}{c|c|c}\omega & a_1,\ldots,a_\omega & \\ 1 & \beta_1,\beta_1' & \\ \omega & \gamma_1,\ldots,\gamma_\omega & \\ 0 & \cdots\cdots & \end{array}\; x, x\right) = {}_{\omega+1}F_\omega\left(\begin{array}{l} a_1,\ldots,a_\omega, \beta_1+\beta_1' ; x \\ \gamma_1,\ldots,\gamma_\omega \end{array}\right); \qquad (8)$$

J. Burchnall and T. W. Chaundy [5] gave an extensive list of expansions of Appell's double hypergeometric functions. In this paper I shall give a number of expansions of Kampe de Feriet's function (1) and show how the results of Burchnall and Chaundy can be deduced, from my general theorems, as particular cases. Burchnall and Chaundy introduced a certain type or differential operator and deduced their results by an application of these operators. Their argument is purely symbolic. My method in deriving the main theorems is straightforward and is based on series derangement. It can give some other expansions which

are not stated in [5]. It may be noted that the parameters and the variables are such that the functions involved exist.

The expansions will be stated and proved in section 2 while the particular cases are deduces in section 3. The following elementary expansion is needed in the proof:

$$ {}_pF_q\left(\begin{matrix}\alpha_1,\ldots,\alpha_p; x+y \\ \\ \beta_1,\ldots,\beta_q\end{matrix}\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\alpha_j; m+n)}{\prod_{j=1}^{q}(\beta_j; n+n)} \frac{x^m \; y^m}{m! \;\; n!} \tag{9} $$

The following results will also be required in the proofs:

Gauss theorem:

$$ {}_2F_1\left(\begin{matrix}\alpha,\beta;1 \\ \\ \gamma\end{matrix}\right) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \quad R(\gamma-\alpha-\beta) > 0; \tag{10} $$

Saalschütz's theorem:

$$ F_2\left(\begin{matrix}\alpha,\beta,\gamma;1 \\ \\ \rho,\sigma\end{matrix}\right) \frac{\Gamma(\rho)\Gamma(1+\alpha-\sigma)\Gamma(1+\beta-\sigma)\Gamma(1+\gamma-\sigma)}{\Gamma(1-\sigma)\Gamma(\rho-\alpha)\Gamma(\rho-\beta)\Gamma(\rho-\gamma)}, \tag{11} $$

where $\rho + \sigma = \alpha + \beta + \gamma + 1$ and if α, β or γ is a negative integer.

2. The main theorems: The expansions to be proved are:

$$\sum_{r=0}^{\infty} \frac{\prod_{j=1}^{\mu} (\alpha_j ; 2r) \prod_{j=1}^{\nu} \{(\beta_j ; r)(\beta_j' ; r)\} (b ; r)(c-b ; r)}{r! \prod_{j=1}^{\rho} (\gamma_j ; 2r) \prod_{j=1}^{\sigma} \{(\delta_j ; r)(\delta_j' ; r)\} \; (c ; r)} (-xy)^r$$

$$\times F \left(\begin{array}{c|l|c} \mu+1 & \alpha_1+2r, \ldots, \alpha_\mu+2r, b+r & \\ \nu & \beta_1+r, \beta_1'+r; \ldots, \beta_\nu+r, \beta_\nu'+r & x, y \\ \rho & \gamma_1+2r, \ldots, \gamma_\rho+2r & \\ \sigma & \delta_1+r, \delta_1'+r; \ldots; \delta_\sigma+r, \delta_\sigma'+r & \end{array} \right)$$

$$= F \left(\begin{array}{c|l|c} \mu+1 & \alpha_1, \ldots, \alpha_\mu, c & \\ \nu+1 & \beta_1\beta_1'; \ldots; \beta_\nu\beta_\nu'; b, b & x, y \\ \rho & \gamma_1, \ldots, \gamma_\rho & \\ \sigma+1 & \delta_1\delta_1'; \ldots; \delta_\delta\delta_\sigma'; c, c & \end{array} \right) \tag{12}$$

where $\mu + \nu \leq \rho + \sigma$.

$$\sum_{r=0}^{\infty} \frac{\prod_{j=1}^{\omega-1} (\alpha_j ; 2r)(\beta_1 ; r)(\beta_1' ; r)(b ; r)(c-b ; r)}{r! \prod_{j=1}^{\omega} (\gamma_j ; 2r)(c ; r)} (-x^2)^r$$

$$\times {}_{\omega+1}F_{\omega} \left(\begin{array}{l} \alpha_1+2r, \ldots, \alpha_{\omega-1}+2r, b+r, \beta+\beta'+2r; x \\ \gamma_1+2r, \ldots, \gamma_\omega+2r \end{array} \right)$$

$$= F \left(\begin{array}{c|l|c} \omega & \alpha_1, \ldots, \alpha_{\omega-1}, c & \\ 2 & \beta_1\beta_1'; b, b & x, x \\ \omega & \gamma_1, \ldots, \gamma_\omega & \\ 1 & c, c & \end{array} \right) \quad , \text{ where } |x| < 1 \tag{13}$$

$$\sum_{r=0}^{\infty} \frac{\prod_{j=1}^{\mu} (a_j ; 2r)(b;r)(c-b;r)}{r! \prod_{j=1}^{\rho} (\gamma_j ; 2r)(c;r)} (-xy)^r$$

$$\times {}_{\mu+1}F_{\rho} \left(\begin{matrix} a_1+2r, \ldots, a_\mu+2r; b+r; x+y \\ \\ \gamma_1+2r, \ldots, \gamma_\rho+2r \end{matrix} \right)$$

$$= F \left(\begin{array}{c|l|c} \mu+1 & a_1, \ldots, a_\mu, c & \\ 1 & b, b & \\ \rho & \gamma_1, \ldots, \gamma_\rho & x, y \\ 1 & c, c & \end{array} \right) , \tag{14}$$

where $\mu \leq \rho$, $|x| + |y| < 1$.

$$\sum_{r=0}^{\infty} \frac{\prod_{j=1}^{\mu} (a_j ; 2r)(b;r)}{r! \prod_{j=1}^{\rho} (\gamma_j ; 2r)} (-xy)^r$$

$$\times {}_{\mu+1}F_{\rho} \left(\begin{matrix} a_1+2r, \ldots, a_\mu+2r, b+r; x+y \\ \\ \gamma_1+2r, \ldots, p_\rho+2r \end{matrix} \right)$$

$$= F \left(\begin{array}{c|l|c} \mu & a_1, \ldots, a_\mu & \\ 1 & b, b & \\ \rho & \gamma_1, \ldots, \gamma_\rho & x, y \\ 0 & \ldots\ldots\ldots\ldots & \end{array} \right) , \tag{15}$$

where $\mu \leq \rho$, $|x| + |y| < 1$.

$$\sum_{r=0}^{\infty} \frac{\prod_{j=1}^{\mu} (\alpha_j;\, 2r)\,(xy)^r}{r!\prod_{j=1}^{\rho} (\gamma_j;\, 2r)(b;\, r)} \; {}_{\mu}F_{\rho}\left(\begin{matrix}\alpha_1+2r, \dots, \alpha_\mu+2r;\, x+y \\ \gamma_1+2r, \dots, \gamma_\rho+2r\end{matrix}\right)$$

$$= F\left(\begin{array}{c|c|c} \mu+1 & \alpha_1, \dots, \alpha_\mu, b & \\ 0 & \cdots\cdots & x, y \\ \rho & \gamma_1, \dots, \gamma_\rho & \\ 1 & b, b & \end{array}\right) , \tag{16}$$

where $\mu \leq \rho + 1$, $|x| + |y| < 1$.

Proof of (12): When $\mu + \nu \leq \rho + \sigma + 1$, then from (1) the left side of (12) is equal to

$$\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\mu} (\alpha_j;\, 2r) \prod_{j=1}^{\nu} \{(\beta_j;\, r)(\beta'_j,\, r)\}(b;\, r)(c-b;\, r)}{r!\prod_{j=1}^{\rho} (\gamma_j;\, 2r) \prod_{j=1}^{\sigma} \{(\delta_j;\, r)(\delta'_j;\, r)\}(c;\, r)} \, (-xy)^r \, \frac{(b+r;\, m+n)}{m!\, n!}$$

$$\times \frac{\prod_{j=1}^{\mu} (\alpha_j+2r;\, m+n) \prod_{j=1}^{\nu} \{(\beta_j+r;\, m)(\beta'_j+r;\, n)\}}{\prod_{j=1}^{\rho} (\gamma_j+2r;\, m+n) \prod_{j=1}^{\sigma} \{(\delta_j+r;\, m)(\delta'_j+r\,;\, m)\}} \, x^m y^n \, .$$

Here write $m = p - r$, $n = q - r$, change the order of summation putting the first summation last, noting that the function (1) is absolutely convergent (see [4], p. 150 and [1], p. 119) and get

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{\prod_{j=1}^{\mu} (\alpha_j;\, p+q) \prod_{j=1}^{\nu} \{(\beta_j,\, p)(\beta'_j;\, q)\}\,(b;\, p+q)}{\prod_{j=1}^{\rho} (\gamma_j;\, p+q) \prod_{j=1}^{\sigma} \{(\delta_j;\, p)(\delta'_j\,;\, q)\}} \, \frac{x^p y^q}{p!\, q!} \times {}_3F_2\left(\begin{matrix}-p, -q, c-b;\, 1 \\ 1-b-p-q\, ,\, c\end{matrix}\right) .$$

Now sum the terminating ${}_3F_2$ by Saalschütz's theorem (11) and the last expression becomes

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{\prod_{j=1}^{\mu}(a_j;p+q)\prod_{j=1}^{\nu}\{(\beta_j;p)(\beta_j';q)\}(c;p+q)(b;p)(b;q)}{\prod_{j=1}^{\rho}(\gamma_j;p+q)\prod_{j=1}^{\sigma}\{(\delta_j;p)(\delta_j';q)\}(c;p)(c;q)} \times \frac{x^p y^q}{p!\,q!}$$

Now apply (1) and obtain the right hand side of (12). For $\mu = \omega - 1$, $\rho = \omega$; (12) in combination with (8) gives (13). Also when $\mu = \rho = 0$; then (12) in combination with (6) gives (14).

Proof of (15): Formula (15) can also be proved in the same way as (12) by applying (9) and using Gauss's theorem (10) instead of Saalschütz's theorem (11). Also formula (15) can be deduced from (14) by letting $c \to \infty$ in (14).

Proof of (16): When $\mu \le \rho + 1$, then from (9) the left hand side of (16) is equal to

$$\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\mu}(a_j;2r)\prod_{j=1}^{\mu}(a_j+2r;m+n)}{r!\prod_{j=1}^{\rho}(\gamma_j;2r)\prod_{j=1}^{\rho}(\gamma_j+2r;m+n)} \frac{(xy)^r}{(b;r)} \frac{x^m y^n}{m!\,n!}\,.$$

Again put $m = p - r$, $n = q - r$, change the order of summation and get

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{\prod_{j=1}^{\mu}(a_j;p+q)\,x^p y^q}{p!\,q!\prod_{j=1}^{\rho}(\gamma_j;p+q)}\, {}_2F_1\left(\begin{matrix}-p,\,-q;1\\ b\end{matrix}\right) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{\prod_{j=1}^{\mu}(a_j;p+q)(b;p+q)}{\prod_{j=1}^{\rho}(\gamma_j;p+q)(b;p)(b;q)} \frac{x^p y^q}{p!\,q!}$$

by Guass's theorem (10). The result now follows from (1). Thus (16) is proved.

3. Further expansions and particular cases: We are now in a position to obtain many expansions included in the list of Burchnall and Chaundy ([5], pp. 253, 254, 255) and other new expansions: Thus in (14) take $\mu = \rho = 1$ and get

$$\sum_{r=0}^{\infty} \frac{(a;2r)(b;r)(c-b;r)}{r!(\gamma;2r)(c;r)}(-xy)^r \; {}_2F_1\left(\begin{matrix} b+r, a+2r; x+y \\ \gamma+2r \end{matrix}\right) = F\left(\begin{matrix} 2 \\ 1 \\ 1 \\ 1 \end{matrix}\middle| \begin{matrix} a, c \\ b, b \\ \gamma \\ c, c \end{matrix} \middle| x, y\right), \tag{17}$$

where $|x| + |y| + |xy| < 1$.

This is a new result which gives equation (43) of the list (5) when $\gamma = c$, namely

$$\sum_{r=0}^{\infty} \frac{(a;2r)(b;r)(c-b;r)}{r!(c;2r)(c;r)}(-xy)^r \times F(a+2r, b+r, c+2r; x+y) = F^{[2]}[a;b,b;c,c;x,y]$$

where $|x| + |y| + |xy$ . .

A particular case of interes.. .s obtained from (14) by taking $y = -x$. Thus if $\mu \leq \rho$

$${}_{2\mu+2}F_{2\rho+1}\left(\begin{matrix} \frac{a_1}{2}, \frac{a_1+1}{2}, \ldots, \frac{a_\mu}{2}, \frac{a_\mu+1}{2}, \; b, c-b; \; 2^{2\mu-2\rho} x^2 \\ \frac{\gamma_1}{2}, \frac{\gamma_1+1}{2}, \ldots, \frac{\gamma_\rho}{2}, \frac{\gamma_{\rho+1}}{2}, \; c \end{matrix}\right) = F\left(\begin{matrix} \mu+1 \\ 1 \\ \rho \\ 1 \end{matrix}\middle| \begin{matrix} a_1, \ldots, a_\mu, c \\ b, b \\ \gamma_1, \ldots, \gamma_\rho \\ c, c \end{matrix} \middle| x, -x\right), \tag{19}$$

where $|x| < 1$ when $\mu = \rho$.

In (19) let $c \to \infty$ and it becomes

$$ {}_{2\mu+1}F_{2\rho}\left(\begin{matrix} \frac{a_1}{2}, \frac{a_1+1}{2}, \dots, \frac{a_\mu}{2}, \frac{a_\mu+1}{2}, b; 2^{2\mu-2\rho}x^2 \\ \\ \frac{\gamma_1}{2}, \frac{\gamma_1+1}{2}, \dots, \frac{\gamma_\rho}{2}, \frac{\gamma_{\rho+1}}{2} \end{matrix}\right) = F\left(\begin{array}{c|c|c} \mu & a_1, \dots, a_\mu & \\ 1 & b, b & \\ \rho & \gamma_1, \dots, \gamma_\rho & x, -x \\ 0 & \dots\dots & \end{array}\right), \quad (20) $$

where $\mu \leq \rho$ and $|x| < 1$ when $\mu = \rho$.

Formula (20) gives a generalization of the formula [6], p. 488 ($\mu = \rho = 1$) namely

$$ {}_3F_2\left(\begin{matrix} \frac{a_1}{2}, \frac{a_1+1}{2}, b; x^2 \\ \\ \frac{\gamma_1}{2}, \frac{\gamma_1+1}{2} \end{matrix}\right) = F^{[1]}[a_1; b, b; \gamma_1; x, -x], \quad (21) $$

where $|x| < 1$.

Again in (17) let $c \to \infty$ and so obtain equation (39) of the list [5] namely

$$ \sum_{r=0}^{\infty} \frac{(a; 2r)(b; r)}{r!(\gamma; r)}(-xy)^r \; {}_2F_1\left(\begin{matrix} a+2r, b+r; x+y \\ \gamma+2r \end{matrix}\right) = F^{[1]}[a; b, b; \gamma; x, y], \quad (22) $$

where $|x| + |y| + |xy| < 1$.

In (16) take $\mu = 2$, $\rho \gamma 1$, and get

$$ \sum_{r=0}^{\infty} \frac{(a_1; 2r)(a_2; 2r)}{r!(\gamma_1; 2r)(b; r)}(xy)^r \; {}_2F_1\left(\begin{matrix} a_1+2r, a_2+2r; x+y \\ \gamma_1 + 2r \end{matrix}\right) = F\left(\begin{array}{c|c|c} 3 & a_1, a_2, b & \\ 0 & \dots\dots & \\ 1 & \gamma_1 & x, y \\ 1 & b, b & \end{array}\right), \quad (23) $$

where $|\sqrt{x}| + |\sqrt{y}| < 1$.

(23) generalizes formula (41) of Burchnall and Chaundy p, 255 of [5] ($b = \gamma_1$), namely

$$\sum_{r=0}^{\infty} \frac{(\alpha_1; 2r)(\alpha_2; 2r)}{r!(\gamma_1; 2r)(\gamma_1; r)} (xy)^r \; {}_2F_1\begin{pmatrix}\alpha_1+2r, \alpha_2+2r; x+y \\ \gamma_1 + 2r\end{pmatrix} = F^{[4]}[\alpha_1, \alpha_2; \gamma, \gamma; x, y]; \quad (24)$$

where $|\sqrt{x}| + |\sqrt{y}| < 1$.

Another particular case of interest is obtained by taking $y = -x$ in (16). Thus we have if $\mu \leq \rho$

$${}_{2\mu}F_{2\rho+1}\begin{pmatrix}\frac{\alpha_1}{2}, \frac{\alpha_1+1}{2}, \ldots, \frac{\alpha_\mu}{2}, \frac{\alpha_{\mu+1}}{2}; \; 2^{2\mu-2\rho} x^2 \\ \frac{\gamma_1}{2}, \frac{\gamma_1+1}{2}, \ldots, \frac{\gamma_\rho}{2}, \frac{\gamma_{\rho+1}}{2}, b\end{pmatrix}$$

$$= F\left(\begin{array}{c|c|c} \mu+1 & \alpha_1, \ldots, \alpha_\mu, b & \\ 0 & \ldots\ldots\ldots & x, -x \\ \rho & \gamma_1, \ldots, \gamma_\rho & \\ 1 & b, b & \end{array}\right), \quad (25)$$

which gives when $\mu = \rho = 1$ and $b = \gamma_1$

$${}_2F_3\begin{pmatrix}\frac{\alpha_1}{2}, \frac{\alpha_1+1}{2}; \; x^2 \\ \frac{\gamma_1}{2}, \frac{\gamma_1+1}{2}, \gamma\end{pmatrix} = F\left(\begin{array}{c|c|c} 1 & \alpha_1 & \\ 0 & \ldots & x, -x \\ 0 & \ldots & \\ 1 & \gamma_1, \gamma_1 & \end{array}\right) \quad (26)$$

where $|x| < 1$.

Also take in (25) $\mu = 2$, $\rho = 1$ with $\gamma_1 = b$ and so obtain the new relation

$$
{}_4F_3\left(\begin{matrix}\frac{\alpha_1}{2}, \frac{\alpha_1+1}{2}, \frac{\alpha_2}{2}, \frac{\alpha_2+1}{2}; 4x^2 \\ \frac{\gamma_1}{2}, \frac{\gamma_1+1}{2}, \gamma_1\end{matrix}\right) = F^{[4]}[\alpha_1, \alpha_2; \gamma_1; x, -x] , \qquad (27)
$$

where $|\sqrt{x}| < \frac{1}{2}$.

Again in (12), take $\mu = \sigma = 1$, $\nu = \rho = 0$, apply (5) and get the following result valid in a suitable region for convergence

$$
\sum_{r=0}^{\infty} \frac{(\alpha_1; 2r)(b;r)(c-b;r)}{r!\,(\delta_1;r)(\delta_1';r)(c;r)} (-xy)^r F^{[4]}[\alpha_1+2r, b+r; \delta_1+r, \delta_1'+r; x, y]
$$

$$
= F\left(\begin{array}{c|c|c} 2 & \alpha_1, c & \\ 1 & b, b & \\ 0 & \cdot\;\cdot\;\cdot & x, y \\ 2 & \delta, \delta_1'; c, c & \end{array}\right) , \qquad (28)
$$

In (28) let $c \to \infty$, apply (3) and so obtain equation (37) of the list [5] namely

$$
\sum_{r=0}^{\infty} \frac{(\alpha_1; 2r)(b;r)}{r!\,(\delta_1;r)(\delta_1';r)} (-xy)^r F^{[4]}[\alpha_1+2r, b+r; \delta_1+r, \delta_1'+r; x, y] = F^{[2]}[\alpha_1; b, b; \delta_1, \delta_1'; x, y], \qquad (29)
$$

where $|x| + 2\,|\sqrt{2xy}| + |y| < 1$.

Also in (28) take $\delta_1 = \delta_1' = b$ and get

$$
\sum_{r=0}^{\infty} \frac{(\alpha_1; 2r)(c-b;r)}{r!\,(b;r)(c;r)} (-xy)^r F^{[4]}[\alpha_1+2r, b+r; b+r, b+r; x, y] = F^{[4]}[\alpha_1, c; c, c; x, y], \qquad (30)
$$

where $|\sqrt{x}| + |\sqrt{y}| < 1$; which is a new relation.

In (12) take $\mu = \rho = 0$, $\nu = \sigma = 1$, apply (3) and get

$$\sum_{r=0}^{\infty} \frac{(\beta_1;r)(\beta_1';r)(b;r)(c-b;r)}{r!(\delta_1;r)(\delta_1';r)(c;r)} (-xy)^r \times F^{[2]}[b+r;\beta_1+r,\beta_1'+r;\delta_1+r,\delta_1'+r;x,y]$$

$$= F\left(\begin{array}{c|c|c} 1 & c & \\ 2 & \beta_1,\beta_1';b,b & x,y \\ 0 & \cdot\;\cdot\;\cdot\;\cdot\;\cdot & \\ 2 & \delta_1\delta_1',c,c & \end{array}\right), \qquad (31)$$

where $|x| + |y| + |xy| < 1$.

In (31) let $c \to \infty$, apply (7) and so obtain equation (27) of the list [5], namely

$$\sum_{r=0}^{\infty} \frac{(\beta_1;r)(\beta_1';r)(b;r)}{r!(\delta_1;r)(\delta_1';r)} (-xy)^r F^{[2]}[b+r;\beta_1+r,\beta_1'+r;\delta_1+r,\delta_1'+r;x,y]$$

$$= {}_2F_1\left(\begin{matrix} b,\ \beta_1;x \\ \\ \delta_1 \end{matrix}\right) {}_2F_1\left(\begin{matrix} b,\beta_1';y \\ \\ \delta_1' \end{matrix}\right), \qquad (32)$$

where $|x| + |y| + |xy| < 1$.

In (12) take $\mu = \sigma = 0$, $\nu = \rho = 1$, apply (2) and get

$$\sum_{r=0}^{\infty} \frac{(\beta_1;r)(\beta_1';r)(b;r)(c-b;r)}{r!(\gamma_1;2r)(c;r)} (-xy)^r F^{[1]}[b+r;\beta_1+r,\beta_1'+r;\gamma_1+2r;x,y]$$

$$= F\left(\begin{array}{c|c|c} 1 & c & \\ 2 & \beta_1,\beta_1';b,b & x,y \\ 1 & \gamma_1 & \\ 1 & c,c & \end{array}\right), \qquad (33)$$

where $|xy| < \frac{1}{2}$.

(33), in virtue of (7), gives for the particular case $\gamma_1 = c$ formula (31) of the list [5] namely

$$\sum_{r=0}^{\infty} \frac{(\beta_1; r)(\beta_1'; r)(b; r)(c-b; r)}{r!\,(c; 2r)(c; r)} (-xy)^r F^{[1]}[b+r; \beta_1+r, \beta_1'+r; c+2r; x, y]$$

$$= {}_2F_1\begin{pmatrix}\beta_1, b; x \\ c\end{pmatrix} {}_2F_1\begin{pmatrix}\beta_1', b; y \\ c\end{pmatrix}, \qquad (34)$$

where $|xy| < \frac{1}{2}$.

When $c \to \infty$, (33) gives if $|xy| < \frac{1}{2}$:

$$\sum_{r=0}^{\infty} \frac{(\beta_1; r)(\beta_1'; r)(b; r)}{r!\,(\gamma_1; 2r)} (-xy)^r F^{[1]}[b+r; \beta_1+r, \beta_1'+r; \gamma_1+2r; x, y]$$

$$= F\left(\begin{array}{c|c|c} 0 & \cdots\cdots & \\ 2 & \beta_1\beta_1'; b, b & x, y \\ 1 & \gamma_1 & \\ 0 & \cdots\cdots & \end{array}\right) = F^{[3]}[\beta_1, \beta_1'; b, b; \gamma; x, y] \; ; \qquad (35)$$

which is formula (33) of the list [5].

In (33) take $\beta = \beta' = c$, apply (2) and get the new relation

$$\sum_{r=0}^{\infty} \frac{(b; r)(c; r)(c-b; r)}{r!\,(\gamma_1; 2r)} (-xy)^r F^{[1]}[b+r; c+r, c+r; \gamma_1+2r; x, y] = F^{[1]}[c; b, b; \gamma_1; x, y], \qquad (36)$$

where $|xy| < \frac{1}{2}$.

In (14) take $\mu = \rho = 0$, apply (3) and get

$$\sum_{r=0}^{\infty} \frac{(b;r)(c-b;r)}{r!\,(c;r)} (-xy)^r (1-x-y)^{-b-r} = F^{[2]}[c;b;b;c,c;x,y] \;, \qquad (37)$$

where $|x| + |y| < 1$.

When $c \to \infty$, then (37) becomes if $|x| < 1$, $|y| < 1$,

$$\sum_{r=0}^{\infty} \frac{(b;r)}{r!} (-xy)^r (1-x-y)^{-b-r} = (1-x-y+xy)^{-b} \;, \qquad (38)$$

which is obvious.

In (16) take $\mu = \rho = 0$, and get if $|x| + |y| < 1$,

$${}_0F_1\left(\begin{matrix} ;xy \\ b \end{matrix}\right) = e^{-x-y} F\left(\begin{array}{c|c|c} 1 & b & \\ 0 & \ldots & \\ & & x, y \\ 0 & \ldots & \\ 1 & b, b & \end{array}\right), \qquad (39)$$

which gives when $y = x$, the known formula ([7], p. 101)

$${}_0F_1\left(\begin{matrix} ;x^2 \\ b \end{matrix}\right) = e^{-2x}\, {}_1F_1\left(\begin{matrix} b-\frac{1}{2}\,;\, 4x \\ 2b-1 \end{matrix}\right) . \qquad (40)$$

In (16) take $\mu = 2$, $\rho = 1$, with $\alpha_1 = \frac{n+m+1}{2}$, $\alpha_2 = \frac{n+m+2}{2}$, $\gamma_1 = n + \frac{3}{2}$;

apply the relation

$$\frac{\Gamma(\frac{n+m+1}{2})\,\Gamma(\frac{n+m+2}{2})}{\Gamma(n+\frac{3}{2})} F\left(\begin{matrix} \frac{n+m+1}{2}, \frac{n+m+2}{2} ; \frac{-1}{z^2} \\ n+\frac{3}{2} \end{matrix}\right) = \frac{i^{n+1}\, z^{n+m+1}}{2^{m-1}(z^2+1)^{\frac{1}{2}m}}\, Q_n^m(iz) ; \qquad (41)$$

where $Q_n^m(z)$ is the associated Legendre function of the second kind; and get

$$\sum_{r=0}^{\infty} (-1)^r \frac{(x+y-1)^{-r}}{2^{2r} r!\,(b;r)} \left(\frac{1}{x}+\frac{1}{y}\right)^{-r} Q_{n+2r}^{m+2r}\left(\frac{-1}{\sqrt{x+y}}\right)$$

$$= (-1)^n \frac{i^{-1-m}\Gamma(n+m+1)}{2^{n+1}\sqrt{\pi}\,\Gamma(n+\frac{3}{2})} F\left(\begin{array}{c|c|c} 3 & \frac{n+m+1}{2}, \frac{n+m+2}{2}, b & \\ 0 & \cdots\cdots\cdots & x, y \\ 1 & n+\frac{3}{2} & \\ 1 & b, b & \end{array}\right), \qquad (42)$$

where $|x + y| < 1$.

Again in (16) take $\mu = 0$, $\rho = 1$, with $\gamma_1 = \nu + 1$, apply the relation

$$\frac{1}{\Gamma(\nu+1)}\ {}_0F_1\left(\begin{matrix} ; z \\ \nu+1 \end{matrix}\right) = \left(\frac{-i\ 2^{\frac{1}{2}}}{z^{\frac{1}{2}}}\right) J_\nu[i(2z)^{\frac{1}{2}}], \qquad (43)$$

where $J_\nu(z)$ is the Bessel function of the first kind; and get

$$\sum_{r=0}^{\infty} \frac{(-1)^r 2^r}{r!\,(b;r)} \left(\frac{x^y}{x+y}\right)^r J_{\nu+2r}\left[i(2x+2y)^{\frac{1}{2}}\right]$$

$$= \frac{(-i)^{-\nu}}{2^{\frac{\nu}{2}}\Gamma(1+\nu)} (x+y)^{\frac{1}{2}\nu} F\left(\begin{array}{c|c|c} 1 & b & \\ 0 & \cdots & x, y \\ 1 & \nu+1 & \\ 1 & b, b & \end{array}\right). \qquad (\ (43)$$

In (14) take $\gamma_1 = c$, apply (7) and so obtain a formula for generalized Whittaker functions (see [8], p. 239) namely

$$\sum_{r=0}^{\infty} \frac{(b;r)(c-b;r)}{r!\,(c;2r)(c;r)} (-xy)^r\ {}_1F_1\left(\begin{matrix} b+r; x+y \\ c+2r \end{matrix}\right) = {}_1F_1\left(\begin{matrix} b; x \\ c \end{matrix}\right)\ {}_1F_1\left(\begin{matrix} b; y \\ c \end{matrix}\right). \qquad (44)$$

Finally in (12) take $\mu = 0$, $\rho = 1$ with $\gamma_1 = c$, apply (7) and get

$$\sum_{r=0}^{\infty} \frac{\prod_{j=1}^{\nu} \{(\beta_j;r)(\beta_j';r)\}(b;r)(c-b;r)}{r!(c;2r)\prod_{j=1}^{\sigma}\{(\delta_j;r)(\delta_j';r)\}(c;r)}(-xy)^r$$

$$\times F\left(\begin{array}{c|l|c} 1 & b+r & \\ \nu & \beta_1+r, \beta_1'+r; \ldots; \beta_\nu+r, \beta_\nu'+r & x, y \\ 1 & c+2r & \\ \sigma & \delta_1+r, \delta_1'+r; \ldots; \delta_\sigma+r, \delta_\sigma'+r & \end{array}\right)$$

$$= {}_{\nu+1}F_{\sigma+1}\left(\begin{matrix}\beta_1, \ldots, \beta_\nu, b; x \\ \\ \delta_1, \ldots, \delta_\sigma, c\end{matrix}\right) {}_{\nu+1}F_{\sigma+1}\left(\begin{matrix}\beta_1', \ldots, \beta_\nu'; b; y \\ \\ \delta_1', \ldots, \delta_\sigma', c\end{matrix}\right), \tag{45}$$

where $\nu \leq \sigma + 1$ and $|x|$, $|y| < 1$ when $\nu < \sigma + 1$.Π

A particular case of interest is obtained from (45) by taking $\nu = 1$, $\sigma = 0$. Thus we have, in virtue of (2)

$$\sum_{r=0}^{\infty} \frac{(\beta_1;r)(\beta_1';r)(b;r)(c-b;r)}{r!(c;2r)(c;r)}(-xy)^r \; F^{[1]}[b+r; \beta_1+r, \beta_1'+r; c+2r; x, y]$$

$$= {}_2F_1\left(\begin{matrix}\beta_1, b; x \\ \\ c\end{matrix}\right) {}_2F_1\left(\begin{matrix}\beta_1', b; y \\ \\ c\end{matrix}\right), \tag{46}$$

which is (34) again.

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